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THE HARMONICS OF A STRETCHED STRING VIBRATING IN A RESISTING MEDIUM

BY C. R. DINES

We shall here consider the harmonics of a string, fastened at both ends and vibrating transversely with small amplitude in a resisting medium, the vibration being due to the tension in the string and to a force which is a linear function of the transverse velocity and the distance of each point of the string from its position of equilibrium.*

It is well known that when a string vibrates in a non-resisting medium, the component notes are in perfect harmony with the fundamental note of the string, but in the medium we are considering this is not always the case.† We shall show, however, in §2, that by a proper choice of the linear function, any two notes may be made harmonic. We shall then consider, in § §3–5, the harmonics of different relative fundamental notes for that linear function, showing that certain notes will have an infinite number of harmonics, others only a finite number, and that, for certain values of the linear function, there is one note which has no harmonics. From these results we shall, in §6, determine the curve into which the string must be initially distorted in order that only harmonics may be obtained for a certain note, and show that, in certain cases, the equation of this curve may be an infinite trigonometric series, in others a finite trigonometric series, and in certain cases the problem has no solution.

The problem reduces to one in the representation of numbers by a binary quadratic form whose type may be any of the three : parabolic, elliptic, or hyperbolic. Thus, aside from the intrinsic importance of the problem, it is of interest also because it is an application of the theory of numbers to mathematical physics. The best known problem of this kind is that of the vibrating drum-head,‡ but here the quadratic form must be elliptic, while, in the case we shall consider, the hyperbolic and parabolic types may also occur.

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* These conditions will be physically realized if we consider the string vibrating in air and tied at each point to its position of equilibrium by elastic bands.

† See Byerly, *Fourier's Series and Spherical Harmonics*, Article 65, page 115.

‡ See Riemann-Weber, *Partielle Differential-Gleichungen der mathematischen Physik*, vol. 2, p. 253. Also Byerly, *Fourier's Series and Spherical Harmonics*, art. 72, p. 129.

1. The equation of vibration in the case under discussion is obtained by adding

$$2k\frac{\partial y}{\partial t}+cy$$

to the left-hand side of the equation of vibration in a non-resisting medium with no force acting other than that due to the tension in the string.† This equation is, then

$$\frac{\partial^2 y}{\partial t^2}-a^2\frac{\partial^2 y}{\partial x^2}+2k\frac{\partial y}{\partial t}+cy=0. \tag{1}$$

The boundary and initial values to be satisfied are

(a) $x=0,\ y=0,$ (b) $x=l,\ y=0,$ (c) $t=0,\ y=f(x)$.

By a method similar to the usual treatment of the case where c is zero* we obtain as normal solutions satisfying the boundary condition (a)

$$y=e^{-kt}\sin\alpha x\cos t\sqrt{\alpha^2a^2-(k^2-c)},$$
$$y=e^{-kt}\sin\alpha x\sin t\sqrt{\alpha^2a^2-(k^2-c)}.$$

Thus the series giving the vibration is seen to be

$$(2)\ y=e^{-kt}\sum_{m=1}^{\infty}\left[A_m\cos t\sqrt{\frac{m^2\pi^2a^2}{l^2}-(k^2-c)}\right.$$
$$\left.+B_m\sin t\sqrt{\frac{m^2\pi^2a^2}{l^2}-(k^2-c)}\right]\left[\sin\frac{m\pi x}{l}\right].$$

Let us now consider the period of any component note. It is clear that the factor e^{-kt} affects merely the amplitude, causing the vibration to die down, and does not affect the period.

Equation (2) may be written

$$y=e^{-kt}\sum_{m=1}^{\infty}C_m\left[\cos\sqrt{\frac{m^2\pi^2a^2}{l^2}-(k^2-c)}\ (t-\tau)\right]\sin\frac{m\pi x}{l}, \tag{3}$$

where

$$A_m=C_m\cos\tau\sqrt{\frac{m^2\pi^2a^2}{l^2}-(k^2-c)},\qquad B_m=C_m\sin\tau\sqrt{\frac{m^2\pi^2a^2}{l^2}-(k^2-c)},$$

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* See Byerly, loc. cit. art. 1, p. 2 (VIII).

† See Byerly, loc. cit. art. 65, p. 113.

$$\tau = \sqrt{\frac{l^2}{m^2\pi^2a^2 - l^2(k^2-c)}}\tan^{-1}\frac{B_m}{A_m}, \qquad C_m^2 = A_m^2 + B_m^2.$$

Now the period T of one complete vibration must be such that

$$\cos\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2-c)}\;(t-\tau) = \cos\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2-c)}\,(t-\tau+T),$$

and therefore

$$\sqrt{\frac{m^2\pi^2a^2}{l^2} - (k^2-c)}\;\; T = 2\pi,$$

or

$$T = \frac{2\pi l}{\sqrt{m^2\pi^2a^2 - l^2(k^2-c)}}$$

The frequency of vibration or the number of vibrations per second is, then,

$$F = \frac{1}{T} = \frac{\sqrt{m^2\pi^2a^2 - l^2(k^2-c)}}{2\pi l}. \tag{4}$$

We shall now give definitions of a few terms to be used in the discussion following. The note given by a single term of the series (3) of index m is called the component note of index m, or more briefly the note m. Absolute harmonics of a fundamental component note are component notes whose frequencies are multiples of the frequency of the fundamental. Two notes are said to be relatively harmonic when their frequencies bear a rational ratio to each other.

2. If $k^2 - c \neq 0$*, we see from (4) that the frequencies of the different component notes are not exact multiples of the frequency of the fundamental, so that we shall not in general have harmonics. We shall show, however, that by a proper choice of k and c any two notes may be made harmonic and shall then discuss the harmonics of different relative fundamental notes for these chosen values of k and c.

Let F_1 and F_2 denote the frequencies of two notes m_1 and m_2. Let

$$\frac{F_1}{F_2} = \frac{h_1}{h_2},$$

where h_1 and h_2 are relatively prime integers. Then, from (4) we have

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* If $k^2 - c = 0$ it will appear from equation (7) that the problem of harmonics is the same as if the string were vibrating in a non-resisting medium.

$$(5)\qquad \frac{F_1}{F_2} = \frac{\sqrt{m_1^2\pi^2a^2 - l^2(k^2 - c)}}{\sqrt{m_2^2\pi^2a^2 - l^2(k^2 - c)}} = \frac{h_1}{h_2}.$$

Now m_1, m_2, h_1, h_2 are fixed; so that (5) can be solved for $k^2 - c$, giving

$$(6)\qquad k^2 - c = \frac{\pi^2a^2}{l^2}\left[\frac{h_1^2m_2^2 - h_2^2m_1^2}{h_1^2 - h_2^2}\right].$$

3. We shall now determine the condition to be satisfied in order that any note m_3 with frequency F_3 may be harmonic with a note m_4 with frequency F_4 for the above value of $k^2 - c$, which rendered m_1 and m_2 harmonic.

In a way similar to that by which we obtained (5), we have

$$(7)\qquad \frac{F_3}{F_4} = \frac{\sqrt{m_3^2\pi^2a^2 - l^2(k^2 - c)}}{\sqrt{m_4^2\pi^2a^2 - l^2(k^2 - c)}} = \frac{h_3}{h_4}.$$

This relation (7) will give all the harmonics of m_3 if we allow h_3 to take on all possible values and determine integral values of h_4 and m_4, such that h_3 and h_4 are relatively prime, which satisfy the relation (7).

Substituting in (7) the value of $k^2 - c$ from (6), we have

$$(8)\qquad \frac{h_3}{h_4} = \frac{\sqrt{m_3^2\pi^2a^2 - \dfrac{l^2\pi^2a^2}{l^2}\left[\dfrac{h_1^2m_2^2 - h_2^2m_1^2}{h_1^2 - h_2^2}\right]}}{\sqrt{m_4^2\pi^2a^2 - \dfrac{l^2\pi^2a^2}{l^2}\left[\dfrac{h_1^2m_2^2 - h_2^2m_1^2}{h_1^2 - h_2^2}\right]}},$$

or

$$(9)\qquad [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]h_4^2 - h_3^2(h_1^2 - h_2^2)m_4^2 = h_3^2(h_2^2m_1^2 - h_1^2m_2^2).$$

4. The problem, then, reduces to one in the representation of a number by a binary quadratic form, namely that of determining for what integral values of h_4 and m_4 the number

$$h_3^2(h_2^2m_1^2 - h_1^2m_2^2)$$

is representable by the form

$$(10)\qquad [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]h_4^2 - h_3^2(h_1^2 - h_2^2)m_4^2.$$

We shall first determine what values of h_3 are possible in order that the equation (9) may have integral solutions. Transposing, we have

$$h_3^2[(h_1^2 - h_2^2)m_4^2 + (h_2^2m_1^2 - h_1^2m_2^2)] = [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]h_4^2.$$

Now h_3^2 is a factor of the left hand side of this equation and so must be a factor of

$$h_4^2[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)]$$

also. But all harmonics will be obtained by taking h_3 and h_4 relatively prime, so that h_3^2 is a factor of

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)].$$

Therefore h_3 is a factor of σ where σ^2 is the greatest square factor in

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)].$$

We shall, however, obtain all solutions for h_4 which are relatively prime to h_3, by substituting for h_3 the value σ, determining all solutions (h_4m_4) and reducing σ/h_4 to lowest terms. For, if $h_4 = u$, $m_4 = m_4$ are solutions of (9) in the form

$$Ah_4^2 - Bh_3^2m_4^2 = Ch_3^2 \tag{11}$$

when h_3 is given the value t, and u is prime to t, it is obvious that $h_4 = u \cdot s$, $h_3 = t \cdot s$, $m_4 = m_4$ are also solutions. But by a proper choice of s, $t \cdot s = \sigma$, so that $h_4 = u$, $h_3 = t$, will always be among the solutions obtained by substituting σ for h_3, solving for h_4 and m_4, and reducing σ/h_4 to lowest terms.

Since we have supposed σ^2 to be the greatest square factor in

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)],$$

we may set

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)] = r\sigma^2,$$

where r contains no square factor. Then, substituting $h_3 = \sigma$, (9) becomes

$$r\sigma^2h_4^2 - \sigma^2(h_1^2 - h_2^2)m_4^2 = \sigma^2(h_2^2m_1^2 - h_1^2m_2^2),$$

or

$$rh_4^2 - (h_1^2 - h_2^2)m_4^2 = h_2^2m_1^2 - h_1^2m_2^2. \tag{12}$$

5. As is usual in problems involving binary quadratic forms, we shall discuss the form (10) under three cases, hyperbolic, elliptic, and parabolic according as the discriminant

$$\Delta = [(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2)](h_1^2 - h_2^2)h_3^2$$

is positive, negative, or zero.

Case I. $\Delta > 0$. *Form hyperbolic.*

a) $$|(h_1^2 - h_2^2)m_3^2| > |h_1^2m_2^2 - h_2^2m_1^2|.$$

b) $$|(h_1^2 - h_2^2)m_3^2| < |h_1^2m_2^2 - h_2^2m_1^2|$$

and $$\operatorname{sgn}(h_1^2m_2^2 - h_2^2m_1^2) \neq \operatorname{sgn}(h_1^2 - h_2^2).^*$$

Case II. $\Delta < 0$. *Form elliptic.*

$$|(h_1^2 - h_2^2)m_3^2| < |h_1^2m_2^2 - h_2^2m_1^2|$$

and $$\operatorname{sgn}(h_1^2m_2^2 - h_2^2m_1^2) = \operatorname{sgn}(h_1^2 - h_2^2).$$

Case III. $\Delta = 0$. *Form parabolic.*†

$$(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2) = 0.$$

We shall now discuss the methods of solution and the number of solutions in each of these cases. It is easily verified that among these solutions will be found $m_4 = m_3$, $h_4 = \sigma$. This solution may be called the identical solution since it gives rise to no new harmonics; for if $m_4 = m_3$, we get from (7), $h_4 = h_3 = \sigma$. This solution will be discarded from the solutions of the equation (9).

CASE I. $\Delta > 0$.

In the discussion of this case we shall consider two sub-cases: first, Δ a perfect square; second, Δ not a perfect square.

Subcase 1. $\Delta = e^2$.

Let $$\delta = D(r, h_1^2 - h_2^2),‡$$

so that $$r = \lambda\delta, \qquad h_1^2 - h_2^2 = \mu\delta.$$

Then (12) becomes

$$\lambda h_4^2 - \mu m_4^2 = \frac{h_2^2m_1^2 - h_1^2m_2^2}{\delta},§$$

and if μ contains a highest square factor ι^2 such that $\mu = \iota^2\mu'$ and μ' contains no square factor, we have

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* The notation $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$ shall be understood to mean that α and β have the same sign.

† Since $h_1 \neq h_2$, $h_1^2 - h_2^2 \neq 0$.

‡ i.e. the greatest common divisor of r and $h_1^2 - h_2^2$.

§ The right hand side of this equation will be integral; for from our definition of r, $(h_1^2 - h_2^2)m_3^2 - (h_1^2m_2^2 - h_2^2m_1^2) \equiv 0 \pmod{\delta}$ and $h_1^2 - h_2^2 \equiv 0 \pmod{\delta}$. Therefore $h_2^2m_1^2 - h_1^2m_2^2 \equiv 0 \pmod{\delta}$.

(14) $$\lambda h_4^2 - \mu'(\iota\, m_4)^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta}.$$

Equation (14) may be written in the form

(15) $$\lambda x^2 - \mu' y^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta},$$

where λ and μ' are relatively prime, and neither contains a square factor. Thus if the discriminant $\Delta = \lambda\mu'$ is a perfect square, $\lambda = \mu' = \pm 1$, and if we denote the different factorizations of $(h_2^2 m_1^2 - h_1^2 m_2^2)/\delta$ by $L_i N_i$, (15) may be written

(16) $$x^2 - y^2 = L_i N_i.$$

Equating the factors of the right hand side to the factors of the left, we have

$$x + y = L_i, \qquad x - y = N_i,$$

or

(17) $$x = \frac{L_i + N_i}{2}, \qquad y = \frac{L_i - N_i}{2}.$$

To reduce the solutions of (16) to those of (12), two cases must be considered.

(A) $\lambda = +1$.

In this case $h_4 = x$, $m_4 = y/\iota$. Thus to obtain integral solutions for m_4, the congruence $y \equiv 0 \pmod{\iota}$ must be satisfied.

(B) $\lambda = -1$.

In this case $h_4 = y$, $m_4 = x/\iota$, and to obtain integral values for m_4 the congruence $x \equiv 0 \pmod{\iota}$ must be satisfied.

It is evident in both of these cases that, if $h_1^2 - h_2^2$ contains no square factor, i.e., if $\iota = 1$, the solutions of (16) are the solutions of (12).

Let us now determine the number of solutions (17). This will evidently be the same as the number of decompositions of $(h_2^2 m_1^2 - h_1^2 m_2^2)/\delta$ into the product of two factors L_i and N_i such that L_i and N_i are of the same parity.

Let $$\frac{h_2^2 m_1^2 - h_1^2 m_2^2}{\delta} = 2^\nu (a_1^\alpha \cdot a_2^\beta \cdots a_p^\kappa),$$

in which the a_k's are different odd prime numbers. Two cases will be considered.

(A) $\nu = 0$.

In this case L_i and N_i are both odd, so that any decomposition of $h_2^2m_1^2 - h_1^2m_2^2)/\delta$ will give integral solutions for x and y. As is well known, the number of different divisors of $(a_1^\alpha \cdot a_2^\beta \cdots a_p^\kappa)$ is

$$(\alpha + 1)\,(\beta + 1) \cdots (\kappa + 1) = N.$$

From this we find immediately the number of different decompositions if we consider two cases.

(a) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta$ is not a perfect square.

There will in this case be just half as many decompositions as there are different divisors, or $N/2$. This will be integral since at least one of the numbers $(\alpha, \beta, \cdots \kappa)$ is odd.

(b) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta = E^2$.

In this case E might be regarded as a double divisor, since $E \cdot E$ is a factorization. Thus the number of different decompositions will be $(N + 1)/2$. However, in the case $L_i = N_i = E$, the y solution is 0, which gives either $h_4 = 0$, or $m_4 = 0$. If $h_4 = 0$, then also $F_4 = 0$, which case will be discussed in §5, Case III. The solution $m_4 = 0$ is trivial, and will be discarded, hereafter, in the discussion. The number of solutions will be taken then as $(N-1)/2$.

(B) $\nu \neq 0$.

(a) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta$ is not a perfect square.

Denote by C the number of decompositions obtained in (A) (a). Then with each L_i of these we can combine $2, 2^2, 2^3, \cdots 2^{\nu-1}$, the remaining even factors $2^{\nu-1}, 2^{\nu-2}, \cdots 2$, respectively, being combined with the corresponding N_i to make it of the same parity as L_i. The number of decompositions such that L_i and N_i are of the same parity is, then

$$(\nu - 1)\,C.$$

(b) $(h_2^2m_1^2 - h_1^2m_2^2)/\delta = E^2$.

Denote by C' the number of solutions obtained in (A) (b), above. Then, reasoning as in (B) (a), the number of solutions is seen to be

$$(\nu - 1)\,C' - \frac{\nu}{2}.^*$$

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* It is evident that ν cannot be odd, if $(h_2^2m_1^2 - h_1^2m_2^2)/\delta$ is a perfect square.

Therefore aside from the trivial solution we have, as the total number of solutions

$$(\nu - 1)\, C'' - \frac{\nu}{2} - 1.$$

To sum up, then, we have seen that, *the number of harmonics of the note* m_3 *will be finite, and an upper limit has been found for their number, this upper limit being exactly equal to the number of harmonics if* $h_1^2 - h_2^2$ *contains no square factor.*

Subcase 2. $\Delta \neq e^2$.

In case the discriminant of the form is not a perfect square we shall use a method of solution frequently employed in representation problems, which we briefly describe as follows:

Let the form $ax^2 + 2bxy + cy^2$ be denoted by (a, b, c) and its discriminant $b^2 - ac$ by Δ as before.

In order that m may be representable by (a, b, c) it is necessary that Δ be a quadratic residue of m, in other words that the congruence

$$n^2 \equiv \Delta (\text{mod } m)$$

be solvable. Let n be a solution of this congruence and p be determined by the equation

$$p = \frac{n^2 - \Delta}{m}.$$

The necessary and sufficient condition that m be representable by (a, b, c) is that (m, n, p) and (a, b, c) belong to the same class. For this it is necessary and sufficient: (1) that the first roots* of the characteristic equations

$$aw^2 + 2bw + c = 0,$$

$$mw^2 + 2nw + p = 0,$$

developed into continued fractions, give rise to periods composed of the same elements in the same order, so that the period of one is given by a circular permutation of the elements of the period of the other; and (2) that the number of elements be odd, or, if even, that the number of elements which in each fraction precede the same element of the period shall be of the same parity.

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* The first root of $ax^2 + 2bx + c = 0$ is defined to be $\frac{-b + \sqrt{b^2 - ac}}{a}$.

Thus if there is a representation we must have, in terms of continued fractions

$$w_{(a,b,c)} = (a_1, a_2, \cdots a_k, A, B \cdots L, A, B \cdots L, \cdots),$$

$$w_{(m,n,p)} = (a'_1, a'_2, \cdots a'_{k'}, A, B \cdots L, A, B \cdots L, \cdots),$$

and the number of elements in the period $(A, B, \cdots L)$ must be odd; or, if even, then $k + k'$ must be even.

Let $$x = (A, B \cdots L, A, B \cdots L, \cdots),$$

and denote the kth and $(k-1)$th convergents of $(a_1, a_2 \cdots a_k)$ by P/Q and P_1/Q_1 respectively; the k'th and $(k'-1)$th convergents of $(a'_1, a'_2 \cdots a'_{k'})$ by P'/Q' and P'_1/Q'_1 respectively; and the last and next to last convergents of the terminating continued fraction $(A, B, \cdots L)$ by p/q and p_1/q_1 respectively.

Let $\begin{pmatrix} P & P_1 \\ Q & Q_1 \end{pmatrix}$ denote the substitution

$$x = Px' + P_1y', \qquad y = Qx' + Q_1y',$$

and set

$$K = \begin{pmatrix} P & P_1 \\ Q & Q_1 \end{pmatrix}, \qquad H = \begin{pmatrix} P' & P'_1 \\ Q' & Q'_1 \end{pmatrix}, \qquad L = \begin{pmatrix} p & p_1 \\ q & q_1 \end{pmatrix}.$$

Let $dx^2 + 2ex + f = 0$ be the quadratic equation with integral coefficients of which x is a root; then, $L^n (n = \cdots -2, -1, 0, 1, 2, \cdots)$ will be automorphic substitutions of the form (d, e, f).* Now K will transform (a, b, c) into (d, e, f); H will transform (m, n, p) into (d, e, f) and therefore KL^nH^{-1} will transform (a, b, c) into (m, n, p). If

$$KL^nH^{-1} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix},$$

then $x = \alpha_n$, $y = \gamma_n$ gives a representation of m, viz., $a\alpha_n^2 + 2b\alpha_n\gamma_n + c\gamma_n^2 = m$. The infinity of substutions KL^nH^{-1} gives an infinity of representations $x = \alpha_n$, $y = \gamma_n$.

If there is one solution, there will exist corresponding values of K, L, and H, and hence an infinite number of solutions. But we have already pointed out that there is one solution, viz., the identical solution $m_4 = m_3$, $h_4 = \sigma$. *The existence of this one solution, then, establishes the existence of an infinite number of solutions.*

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* Called the reduced form of (a, b, c).

Applying this result, we see in the present case, that, *if* m_1 *and* m_2 *are harmonic, then with every tone* m_3 *is associated an infinite system of harmonics.*

CASE II. $\Delta < 0$.

As in (2), Case I, we determine the form (m, n, p). In the present case, however, the reduced form, which we shall denote by (A, B, C), is such that the relation between the coefficients is

$$C > A \geqq 2|B|.$$

From the Theory of Quadratic Forms, we know that any form may be transformed into the reduced form of its class by repeated applications of the modular substitutions

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.^*$$

As in (2) of Case I, we determine the substitution KLH^{-1}, L being an automorphic substitution of (A, B, C), K the product of the transformations required to bring (a, b, c) into (A, B, C) and H^{-1} the inverse of the product of the transformations required to bring (m, n, p) into (A, B, C). But in this case there are not an infinite number of automorphic substitutions of the reduced form (A, B, C). Their number depends on the number of integral solutions (t, u) of the Pellian equation $t^2 - \Delta u^2 = \sigma^2$, where $\sigma = D(A, 2B, C)$.

In any case there are two automorphs. If $-4\Delta = 3\sigma^2$, there will be four others, while if $-4\Delta = 4\sigma^2$, there will be two others. If $b = 0$, there cannot be six automorphs of (a, b, c), and there can be four automorphs only when $a = c$. For, if $b = 0$ the coefficients of (A, B, C) will be the same as those of (a, b, c), in the same or the reverse order. There, cannot, then, be six automorphic substitutions of the reduced form, for $-4\Delta \neq 3\sigma^2$, since $4ac \neq 3[D(a, c)]^2$. There can be four automorphs only when $4ac = 4[D(a, c)]$, that is, when $a = c$.

Thus the reduced form of the class to which the form (10) under discussion belongs, will have two automorphic substitutions except when $r = h_2^2 - h_1^2$, in which case there are four. We shall consider these cases separately.

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* See Cahen, *Theorie des nombres*, p. 222.

If $r = h_2^2 - h_1^2$, (12) reduces to

$$h_4^2 + m_4^2 = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{h_2^2 - h_1^2}.$$

The right hand side of this equation will be integral, for, from our definition of r,

$$[(h_1^2 - h_2^2)m_3^2 - (h_1^2 m_2^2 - h_2^2 m_1^2)] \equiv 0 \pmod{r};$$

and if $r = h_2^2 - h_1^2$, then

$$h_2^2 m_1^2 - h_1^2 m_2^2 \equiv 0 \pmod{[h_1^2 - h_2^2]}.$$

Thus equation (12) will reduce to

$$x^2 + y^2 = k, \tag{18}$$

where $x = h_4$, $y = m_4$ and

$$k = \frac{h_2^2 m_1^2 - h_1^2 m_2^2}{h_2^2 - h_1^2}.$$

This form has been so exhaustively studied that a detailed discussion of it is unnecessary here. We shall merely state that, when k is representable by $x^2 + y^2$, there will, if μ is the number of different prime factors of k, be 2^μ representations if (x, y) and $(-x, -y)$ are regarded as the same representation, but (x, y) and (y, x) as different representations.*

One of these will be the identical solution and the other will be $h_4 = m_3$, $m_4 = h_3$, which might be called the inverse of the identical solution. But this latter solution gives rise to no new harmonics, since from (8), if $m_3 \gtreqless m_4$, then $h_3 \gtreqless h_4$. Thus if $h_4 = m_3$, $m_4 = h_3$, then $h_4 = h_3 = m_4 = m_3$. This solution is an identical solution and will be discarded. Thus we shall have in this case $2^\mu - 2$ harmonics, where μ is the number of different prime factors of

$$\frac{h_2^2 m_1^2 - h_1^2 m_2^2}{h_2^2 - h_1^2}.$$

If r is not equal to $h_2^2 - h_1^2$, a detailed discussion is necessary. Let R be the number of solutions of the congruence $n^2 \equiv \Delta \pmod{m}$. Then for each of these R solutions, there are always two automorphic substitutions giving altogether $2R$ representations. We shall consider now (x, y), $(-x, -y)$, $(-x, y)$, $(x, -y)$ as the same solution, and we shall have $\frac{1}{2}R$ as the number of solutions. Discarding the identical solution, we have $\frac{1}{2}R - 1$ as the number of harmonics. If k is prime there will be no harmonics since the congruence $x^2 \equiv a (\text{mod } p)$, where p is a prime, has only two solutions.

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* Dedekind-Dirichlet, *Vorlesungen über Zahlentheorie*, p. 68.

Thus we see, that, in the present case, *if* m_1 *and* m_2 *are harmonics, then with every tone* m_3 *is associated a finite number of harmonics.*

Case III. $\Delta = 0$.

In this case, we have $(h_1^2 - h_2^2)m_3^2 = h_1^2 m_2^2 - h_2^2 m_1^2$,

or
$$m_3^2 = \frac{h_1^2 m_2^2 - h_2^2 m_1^2}{h_1^2 - h_2^2}.$$

Also from (9),
$$m_4^2 = \frac{h_1^2 m_2^2 - h_2^2 m_1^2}{h_1^2 - h_2^2},$$

that is
$$m_3 = m_4,$$

and the note m_3 has no harmonics.

In fact the note m_3 has an infinite period as is seen from the equation (4). It is obvious from (3) that the vibration of the string in this case is not periodic, for the only function of t entering into the equation of vibration is e^{-kt}.

We shall outline a method for determining what values of m_3, h_1, h_2, will give us this case when m_1 and m_2 are chosen.

We have
$$\frac{h_1^2}{h_2^2} = \frac{m_1^2 - m_3^2}{m_2^2 - m_3^2}.$$

Now h_1 prime to h_2 will give us all the harmonics of m_3; thus in order that the above relation may hold, a necessary condition is

$$m_1^2 - m_3^2 = a h_1^2, \qquad m_2^2 - m_3^2 = a h_2^2 \quad (a, \text{ integral}),$$

or
$$h_2^2 - h_1^2 = \frac{m_2^2 - m_1^2}{a}.$$

Since h_1 and h_2 are integral, a is limited in value to the factors of $m_2^2 - m_1^2$.

Let
$$\frac{m_2^2 - m_1^2}{a} = L_i N_i.$$

Then $$h_2^2 - h_1^2 = L_i N_i,$$

and $$h_2 = \frac{L_i + N_i}{2}, \quad h_1 = \frac{L_i - N_i}{2}.$$

If these give integral values for h_1 and h_2, we determine m_3 by the relation

$$m_3^2 = m_1^2 - ah_1^2,$$

and if m_3 is integral, the relation necessary for this case is obtained. If we consider h_1 always smaller than h_2, i.e., m_1 less than m_2, we need give a only positive values, noting that ah_1^2 cannot be greater than m_1^2. It is evident that, if m_1, m_2, m_3 have values satisfying this relation, then their multiples also are solutions for the same values of h_1 and h_2.

By the method here outlined, I have found that the only values of m_1 and m_2 such that $m_1 < m_2 < 10$, for which the form (10) is parabolic, together with the corresponding values of h_1, h_2, m_3, are:

$$m_1 = 3, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 3, \quad m_3 = 2,$$

$$m_1 = 2, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 4, \quad m_3 = 1.$$

We have, then, shown in §5 that: *if the form* (10) *is hyperbolic, the note* m_3 *may have either a finite or an infinite number of harmonics; if* (10) *is elliptic there are only a finite number of harmonics of* m_3*; and if* (10) *is parabolic there are no harmonics of* m_3.

6. The results that we have obtained in the preceding sections enable us to determine the curve $y = f(x)$ into which the string must be initially distorted in order to obtain a musical note, that is, so that all the component notes may be harmonics.

Putting $t = 0$ in (2), we obtain

$$f(x) = e^{-kt} \sum_{m=1}^{\infty} A_m \sin \frac{m\pi x}{l}.$$

Now, if we wish the note m_k, not harmonic, to drop out, we must have $A_k = 0$ in the equation of the initial curve.

Thus we see that, *in certain cases, we shall have as the equation of the initial curve, an infinite trigonometric series, in others, a finite trigonometric series, and in others, there is no solution for the problem.*

7. We shall now give a few examples, illustrative of the general discussion. In these we shall first choose values for h_1, h_2, m_1, m_2, thus fixing the medium in which the string vibrates. Then we shall consider the harmonics of the different notes of the string, when vibrating in that medium.

Example 1. $m_1 = 2, \quad m_2 = 3, \quad h_1 = 1, \quad h_2 = 2.$

The parabolic case does not occur.

The elliptic case occurs only for $m_3 = 1$.

Thus the form is hyperbolic for $m_3 > 1$. There is no value of m_3 for which the discriminant of the form is a perfect square.

(*a*) $m_3 = 1. \qquad h_3 = \sigma = 2.$

Equation (9) reduces to

$$h_4^2 + 3m_4^2 = 7.$$

Here we have $\Delta = -3$, and the congruence

$$n^2 \equiv -3 \pmod{7}$$

has only two incongruent solutions. Thus, $R = 2$, and there is only one solution, viz., the identical solution

$$h_4 = h_3 = 2, \qquad m_4 = m_3 = 1.$$

(*b*) If $m_3 > 1$, the note m_3 has an infinite number of harmonics. We shall consider the case for $m_3 = 4$. We have here

$$m_3 = 4,$$
$$h_3 = \sigma = 1,$$
$$3m_4^2 - 41h_4^2 = 7,$$
$$\Delta = 123,$$

and the congruence,

$$n^2 \equiv 123 \pmod{7} \equiv 4 \pmod{7},$$

has the incongruent solutions $n = \pm 2$. Since a change in sign of n changes only the signs of the variables, we shall use only $n = +2$. This gives

$$p = -17,$$
$$(m, n, p) = (7, 2, -17),$$
$$w_{(3,0,-41)} = (3; 1, 2, 3, 2, 1, 6; 1, 2, 3, 2, 1, 6; \cdots)$$
$$w_{(7,2,-17)} = (1, 3, 2, 1, 6; 1, 2, 3, 2, 1, 6; 1, 2, 3, 2, 1, 6; \cdots)$$

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

$$H^{-1} = \begin{pmatrix} 10 & -13 \\ -67 & 87 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 221 & 33 \\ 154 & 23 \end{pmatrix}$$

$$KH^{-1} = \begin{pmatrix} -37 & 48 \\ 10 & -13 \end{pmatrix} \qquad m_4 = 34, \quad h_4 = 10.$$

$$KL_1H^{-1} = \begin{pmatrix} -4 & -7 \\ -1 & -2 \end{pmatrix} \qquad m_4 = 4, \quad h_4 = 1.$$

$$KL_1^{-1}H^{-1} = \begin{pmatrix} 9024 & 11719 \\ 2441 & 3170 \end{pmatrix} \quad m_4 = 9024, \quad h_4 = 2441.$$

The second pair of solutions is the identical solution. From the first pair, we see that the ratio of the frequency of the thirty-fourth note to that of the fourth is 1 : 10. From the third pair, we see that the ratio of the frequency of the nine-thousand-and-twenty-fourth note to that of the fourth is 1 : 2441. By using the infinite number of L's, we may obtain an infinite number of harmonics.

Thus, if the second and third notes are harmonic, with the ratio of their frequencies 1 : 2, the first note has no harmonics and any other note than the first has an infinite number of harmonics.

Example 2. $m_1 = 1, \quad m_2 = 5, \quad h_1 = 1, \quad h_2 = 2.$

The form (10) is hyperbolic for every value of m_3. The only value of m_3 for which Δ is a perfect square is $m_3 = 3$. The only solution of equation (9) for $m_3 = 3$ is the identical solution.

Thus, if the first and fifth notes are harmonics, with the ratio of their frequencies 1 : 2, the third note has no harmonics and any note other than the third has an infinite number of harmonics.

Example 3. $m_1 = 4, \quad m_2 = 9, \quad h_1 = 2, \quad h_2 = 3.$

The form (10) is hyperbolic for every value of m_3. The only value of m_3 for which Δ is a perfect square is $m_3 = 8$, and for $m_3 = 8$ the only solution of equation (9) is the identical solution.

Thus, if the fourth and ninth notes are harmonics, with frequencies in the ratio 2 : 3, the eighth note has no harmonics and any other note has an infinite number of harmonics.

Example 4. $m_1 = 2, \quad m_2 = 3, \quad h_1 = 1, \quad h_2 = 4.$

The form (10) is elliptic for $m_3 = 1$, and hyperbolic for $m_3 > 1$. The discriminant of the form cannot be a perfect square. The only solution of equation (9) for $m_3 = 1$ is the identical solution.

Thus, if the second and third notes are harmonics, with the ratio of their frequencies 1 : 4, the first note has no harmonics and any note other than the first has an infinite number of harmonics.

Example 5. $m_1 = 2, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 4.$

The form (10) is parabolic for $m_3 = 1$, and hyperbolic for $m_3 > 1$. The discriminant cannot be a perfect square.

Thus, if the second and seventh notes are harmonics, with the ratio of their frequencies 1 : 4, the first note has no harmonics, and any other note has an infinite number of harmonics.

Example 6. $m_1 = 1, \quad m_2 = 8, \quad h_1 = 4, \quad h_2 = 5.$

The form (10) is hyperbolic for every value of m_3, and the discriminant is a perfect square for $m_3 = 17$ or 55. The fifty-fifth note is the only harmonic of the seventeenth note and vice versa. These two notes, then, form a system of harmonics. With any other note of the string is associated an infinite system of harmonics.

Example 7. $m_1 = 3, \quad m_2 = 7, \quad h_1 = 1, \quad h_2 = 3.$

The form (10) is elliptic for $m_3 = 1$, parabolic for $m_3 = 2$, and hyperbolic for $m_3 > 2$. The discriminant of the form cannot be a perfect square. The only solution of equation (9) for $m_3 = 1$ is the identical solution.

Thus, we see that, if the third and seventh notes are harmonics with the ratio of their frequencies 1 : 3, the first and second notes have no harmonics, and any other note has an infinite number of harmonics.